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# The Wiener-Hopf Formulation of the Penetrable Wedge Problem.

## Part III: The skew incidence case

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**Abstract** *In earlier papers the author obtained the semi-analytical solution of the generalized Wiener-Hopf equations that formulate the diffraction of a plane wave by a dielectric wedge at normal incidence. This solution permits the separating, recognizing and understanding of the different components of the dielectric wedge field: reflected and refracted plane waves, surface waves, lateral waves, diffraction coefficients. This paper considers the general case of a plane wave impinging on an arbitrary penetrable wedge at skew incidence. Comparing the author's previous findings with the skew incident case, no new conceptual and numerical difficulties are present. However, the involved algebraic manipulations become very cumbersome and to handle them the author resorted to a MATHEMATICA computer program.*

**Keywords** dielectric wedge, diffraction wedge, Wiener-Hopf technique

### 1. Introduction

The Wiener-Hopf (WH) technique is a very powerful mathematical tool for solving field problems. In this technique, these problems are rephrased in terms of functional equations that involve the Laplace transforms of suitable unknowns (spectra). In the presence of angular regions, the WH equations (WHE) assume a form we call Generalized Wiener Hopf equations (GWHE), (Daniele, 2001). In the case of impenetrable wedges a transformation (Daniele, 2001; 2003a) reduces the solution of the GWHE to a classical factorization problem. Several canonical problems involving impenetrable wedges admit an exact factorization and consequently an exact solution (Daniele, 2003a; 2004b). Conversely, regarding penetrable wedges, closed form solutions of the GWHE are amenable only to very special cases. However, a semi-analytical solution is always possible. In particular we can resort to the Fredholm factorization (Daniele, 2004a) which is a technique that reduces the WHE to the solution of Fredholm integral equations of second kind (FIE). We experienced that some expedients, such as that of deforming the integration line of the Fredholm integral (Daniele, 2004a), make the Fredholm factorization very efficient.

The solution of the GWHE for penetrable wedges has required several years of study (Daniele (2004b, 2005, 2009, 2010, 2011; Daniele&Lombardi, 2011)). The present work

completes these studies by considering the more general case of a plane wave impinging on an arbitrary penetrable isotropic wedge at skew incidence.

This paper is organized as follows. Section 2 reports the GWHE of the wedge problem considered in fig.1. These equations are immediately deduced from general equations that relate the spectra for two different values of the observation angles (Daniele 2001,2003, 2004b, 2005, 2010). For the case of normal incidence, the reduction of the GWHE equations to suitable FIE has been accomplished in Daniele (2005, 2010, 2011). The skew incidence requires some modifications that are discussed in section 3.

The numerical solution of the FIE provides accurate representations of the Wiener Hopf (WH) unknowns only in certain regions of the spectral domain. Since these analytical elements are not sufficient to obtain the field on the  $x$  axis (fig.1), a process of analytical continuation is necessary. For the normal incidence, the analytical continuation has been considered in Daniele (2009,11). In this paper, section 4 reports the analytical continuation in the case of skew incidence. The analytical continuations complete the determination of the axial spectra of the plus Wiener-Hopf unknowns (the spectra in the direction  $\varphi = 0$  and  $\varphi = \pi$ ). The expressions of the spectra for every direction  $\varphi$  are reported in section 5. They have been obtained by using the rotating waves theory (Daniele, 2003b). The paper concludes with three Appendices. Appendix A provides some details of the deduction of the FIE. Appendix B reports the explicit expressions of several functions defined in the paper and Appendix C presents a discussion of the compactness of the kernels involved in the FIE.

The involved algebraic manipulations required in this work are very cumbersome and to handle them the techniques described in this paper have been combined in a MATHEMATICA computer program. This program is reported in Daniele (2009).

For the normal case, to ascertain the validity of the proposed method a comparison with the very few published data and a study of convergence of the proposed numerical solution of the FIE have been reported in Daniele (2010,2011) and Daniele & Lombardi (2011). In particular Daniele & Lombardi (2011) present several test cases that provide the geometrical Optics (GO) field and the UTD components for the total far field. For the skew incidence case, a similar study will be undertaken in a future paper.

## 2. The generalized Wiener-Hopf equations of the problem

We consider the diffraction problem indicated in fig.1 where the harmonic electromagnetic fields  $\mathbf{E}\exp(j\omega t)$  and  $\mathbf{H}\exp(j\omega t)$  are present. The penetrable wedge (regions 3 and 4) has permittivity  $\varepsilon_1 = \varepsilon_r \varepsilon_o$  and permeability  $\mu_1 = \mu_r \mu_o$  and is immersed in the free space (regions 1 and 2) having permittivity  $\varepsilon = \varepsilon_o$  and permeability  $\mu = \mu_o$ . The difficulty of this problem has given rise to many techniques for the solution (Lewin & Sreenivasiah, 1979; Vasil'ev & Solodukhov, 1974; Rawlins, 1977,1999; Berntsen, 1983; Kim & Ra and Shin, 1991a,b; Budaev, 1995; Crosille & Lebeau, 1999; Salem, Kamel & Osipov, 2006). The WH technique of the penetrable wedge problem constitutes a method that is based on the solution of suitable GWHE. In this new approach, by assuming the polar coordinates  $\rho, \varphi, z$ , the following Laplace transforms are introduced:

$$V_{z+}(\eta, \varphi) = \int_0^\infty E_z(\rho, \varphi) e^{j\eta\rho} d\rho, \quad I_{\rho+}(\eta, \varphi) = \int_0^\infty H_\rho(\rho, \varphi) e^{j\eta\rho} d\rho \quad (1)$$

In definitions (1), the subscript  $+$  indicates plus functions, i.e. functions having convergence half-planes that are upper half-planes in the  $\eta$ -plane. We also define minus functions the functions having convergence half-planes that are lower half-planes in the  $\eta$ -plane.

$$E_z^i = E_o e^{j\tau_o \rho \cos(\varphi - \varphi_o)} e^{-j\alpha_o z} \quad H_z^i = H_o e^{j\tau_o \rho \cos(\varphi - \varphi_o)} e^{-j\alpha_o z} \quad (2)$$

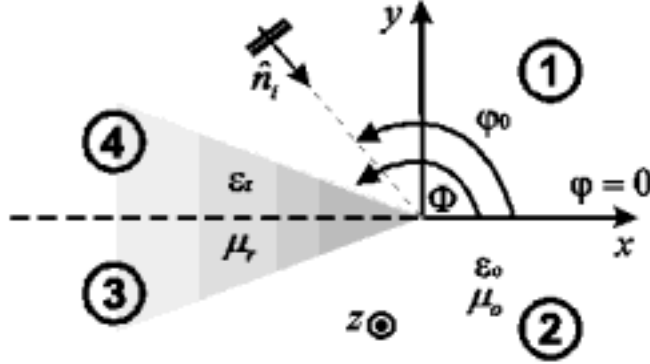


Fig.1 : The dielectric wedge problem

$$\begin{aligned}
&= -n V_{z^+}(-m, \Phi) - \frac{\tau_o^2}{\omega \varepsilon} I_{\rho^+}(-m, \Phi) + \frac{\alpha_o m}{\omega \varepsilon} I_{z^+}(-m, \Phi) \\
&\quad \xi I_{z^+}(\eta, 0) + \frac{\tau_o^2}{\omega \mu} V_{\rho^+}(\eta, 0) + \frac{\alpha_o \eta}{\omega \mu} V_{z^+}(\eta, 0) = \\
&= -n I_{z^+}(-m, \Phi) + \frac{\tau_o^2}{\omega \mu} V_{\rho^+}(-m, \Phi) - \frac{\alpha_o m}{\omega \mu} V_{z^+}(-m, \Phi)
\end{aligned} \tag{4}$$

$$\xi V_{z+}(\eta, 0) + \frac{\tau_o^2}{\omega \varepsilon} I_{\rho+}(\eta, 0) + \frac{\alpha_o \eta}{\omega \varepsilon} I_{z+}(\eta, 0) = \quad (5)$$

$$= -n V_{z+}(-m, -\Phi) + \frac{\tau_o^2}{\omega \varepsilon} I_{\rho+}(-m, -\Phi) - \frac{\alpha_o m}{\omega \varepsilon} I_{z+}(-m, -\Phi)$$

$$\xi I_{z+}(\eta, 0) - \frac{\tau_o^2}{\omega \mu} V_{\rho+}(\eta, 0) - \frac{\alpha_o \eta}{\omega \mu} V_{z+}(\eta, 0) = \quad (6)$$

$$= -n I_{z+}(-m, -\Phi) - \frac{\tau_o^2}{\omega \mu} V_{\rho+}(-m, -\Phi) + \frac{\alpha_o m}{\omega \mu} V_{z+}(-m, -\Phi)$$

$$-\xi_1 V_{z+}(\eta_1, -\pi) + \frac{\tau_1^2}{\omega \varepsilon_1} I_{\rho+}(\eta_1, -\pi) + \frac{\alpha_1 \eta_1}{\omega \varepsilon_1} I_{z+}(\eta_1, -\pi) = \quad (7)$$

$$= n_1 V_{z+}(-m_1, -\Phi) + \frac{\tau_1^2}{\omega \varepsilon_1} I_{\rho+}(-m_1, -\Phi) - \frac{\alpha_1 m_1}{\omega \varepsilon_1} I_{z+}(-m_1, -\Phi)$$

$$-\xi_1 I_{z+}(\eta_1, -\pi) - \frac{\tau_1^2}{\omega \mu_1} V_{\rho+}(\eta_1, -\pi) - \frac{\alpha_1 \eta_1}{\omega \mu_1} V_{z+}(\eta_1, -\pi) = \quad (8)$$

$$= n_1 I_{z+}(-m_1, -\Phi) - \frac{\tau_1^2}{\omega \mu_1} V_{\rho+}(-m_1, -\Phi) + \frac{\alpha_1 m_1}{\omega \mu_1} V_{z+}(-m_1, -\Phi)$$

$$\xi_1 V_{z+}(\eta_1, \pi) + \frac{\tau_1^2}{\omega \varepsilon_1} I_{\rho+}(\eta_1, \pi) + \frac{\alpha_1 \eta_1}{\omega \varepsilon_1} I_{z+}(\eta_1, \pi) = \quad (9)$$

$$= -n_1 V_{z+}(-m_1, \Phi) + \frac{\tau_1^2}{\omega \varepsilon_1} I_{\rho+}(-m_1, \Phi) - \frac{\alpha_1 m_1}{\omega \varepsilon_1} I_{z+}(-m_1, \Phi)$$

$$-\xi_1 I_{z+}(\eta_1, \pi) + \frac{\tau_1^2}{\omega \mu_1} V_{\rho+}(\eta_1, \pi) + \frac{\alpha_1 \eta_1}{\omega \mu_1} V_{z+}(\eta_1, \pi) = \quad (10)$$

$$= n_1 I_{z+}(-m_1, \Phi) + \frac{\tau_1^2}{\omega \mu_1} V_{\rho+}(-m_1, \Phi) - \frac{\alpha_1 m_1}{\omega \mu_1} V_{z+}(-m_1, \Phi)$$

where  $\alpha_1 = k_1 \sin \beta_1 = \alpha_o$ ,  $\eta_1 = \eta_o$ ,  $\varepsilon_1 = \varepsilon_r \varepsilon_o$ ,  $\mu_1 = \mu_r \mu_o$ ,  $k_1 = \omega \sqrt{\mu_1 \varepsilon_1} = \sqrt{\varepsilon_r \mu_r} k_o$ ,

$\tau_o = \sqrt{k_o^2 - \alpha_o^2}$ ,  $\tau_1 = \sqrt{k_1^2 - \alpha_1^2} = \lambda_e \tau_o$ ,  $\lambda_e = \sqrt{1 + (\varepsilon_r \mu_r - 1) \csc^2 \beta}$

$\xi = \xi(\eta) = \sqrt{\tau_o^2 - \eta^2}$  with the branch  $\xi(0) = \tau_o$ ,  $\xi_1 = \xi_1(\eta_1) = \sqrt{\tau_1^2 - \eta_1^2}$  with the branch

$\xi_1(0) = \tau_1$ ,  $m = m(\eta) = -\eta \cos \Phi + \xi \sin \Phi$ ,  $m_1 = m_1(\eta_1) = -\eta_1 \cos \Phi_1 + \xi_1 \sin \Phi_1$

$n = n(\eta) = -\xi \cos \Phi - \eta \sin \Phi$ ,  $n_1 = n_1(\eta_1) = -\xi_1 \cos \Phi_1 - \eta_1 \sin \Phi_1$

Summing (3), (5) and subtracting (7) by (9) we get:

$$2\xi V_{z+}(\eta, 0) = -n s_{vz+}(-m) - \frac{\tau_o^2}{\omega \varepsilon} d_{i\rho+}(-m) + \frac{\alpha_o m}{\omega \varepsilon} d_{iz+}(-m) \quad (11)$$

$$2\xi_1 V_{z+}(\eta, -\pi) = -n_1 s_{vz+}(-m_1) + \frac{\tau_1^2}{\omega \varepsilon_1} d_{i\rho+}(-m_1) - \frac{\alpha_o m_1}{\omega \varepsilon_1} d_{iz+}(-m_1) \quad (12)$$

where:

$$s_{ab+}(-c) = a_{b+}(-c, \Phi) + a_{b+}(-c, -\Phi), \quad d_{ab+}(-c) = a_{b+}(-c, \Phi) - a_{b+}(-c, -\Phi)$$

$a$  stands  $V, I$ ;  $b$  stands  $z, \rho$ ;  $c$  stands  $m, m_1$ ;

Similar equations can be obtained by subtracting (3), (5) and summing (7), (9); by subtracting (8), (10) and summing (4), (6); by summing (8), (10) and subtracting (4), (6).

By substituting  $k_o$  with  $\tau_o$  and  $k_1$  with  $\tau_1$ , we can once again introduce the normalizations considered in Daniele (2010). They derive from the factorization of the scalars  $\xi = \xi_- \xi_+, \xi_1 = \xi_{1-} \xi_{1+}, n = n_- n_+, n_1 = n_{1-} n_{1+}$  and, for the skew incidence case, yield four uncoupled systems ( $i = 1, 3, 5, 7$ ):

$$Y_{(i)+}(\eta) = X_{(i)+}(-m) - \frac{\xi_-}{n_+} X_{(i+1)+}(-m) \quad (13)$$

$$Y_{(i+1)+}(\eta) = \dot{X}_{(i)+}(-m_1) + \frac{\xi_{1-}}{n_{1+}} \dot{X}_{(i+1)+}(-m_1) \quad (14)$$

The plus functions  $Y_{(i)+}(\eta)$  and the minus functions  $X_{(i)+}(-m), \dot{X}_{(i)+}(-m_1)$  are auxiliary functions. In order to simplify the expressions that relate them to the original W-H unknowns, it is convenient to introduce the mappings:

$$\eta = -\tau_o \cos w = -\tau_1 \cos w_1 \quad (15)$$

In the planes  $w$  and  $w_1$  we get (Daniele, 2009):

$$\hat{V}_{\rho+}(w) = -\frac{1}{2} \cos \frac{\pi w}{2\Phi} \left( \cot w \cot \beta \sin \frac{\pi w}{2\Phi} \hat{Y}_1(w) + Z_o \csc \beta \hat{Y}_7(w) \right) \quad (16)$$

$$\hat{V}_{z+}(w) = -\frac{1}{4} \csc w \sin \frac{\pi w}{\Phi} \hat{Y}_1(w) \quad (17)$$

$$\hat{I}_{\rho+}(w) = \frac{1}{2Z_o} \cos \frac{\pi w}{2\Phi} \left( -\csc \beta \hat{Y}_3(w) + Z_o \cot w \cot \beta \sin \frac{\pi w}{2\Phi} \hat{Y}_5(w) \right) \quad (18)$$

$$\hat{I}_{z+}(w) = \frac{1}{4} \csc w \sin \frac{\pi w}{\Phi} \hat{Y}_5(w) \quad (19)$$

$$\hat{V}_{\rho\pi+}(w_1) = \frac{1}{2\lambda_e} \cos \frac{\pi w_1}{2\Phi_1} \left( -\cot w_1 \cot \beta \sin \frac{\pi w_1}{2\Phi_1} \hat{Y}_2(w_1) + Z_o \csc \beta \hat{Y}_8(w_1) \right) \quad (20)$$

$$\hat{V}_{z\pi+}(w_1) = -\frac{1}{4} \csc w_1 \sin \frac{\pi w_1}{\Phi_1} \hat{Y}_2(w_1) \quad (21)$$

$$\hat{I}_{\rho\pi+}(w_1) = \frac{1}{2Z_o \lambda_e} \cos \frac{\pi w_1}{2\Phi_1} \left( \varepsilon_r \csc \beta \hat{Y}_4(w_1) + Z_o \cot w_1 \cot \beta \sin \frac{\pi w_1}{2\Phi_1} \hat{Y}_6(w_1) \right) \quad (22)$$

$$\hat{I}_{z\pi+}(w_1) = \frac{1}{4} \csc w_1 \sin \frac{\pi w_1}{\Phi_1} \hat{Y}_6(w_1) \quad (23)$$

where  $Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}}$ ;  $Z_1 = \sqrt{\frac{\mu_r}{\varepsilon_r}} Z_o$ ;  $\hat{Y}_i(w) = Y_{(i)+}(\eta), (i = 1, 3, 5, 7); \hat{Y}_i(w_1) = Y_{(i)+}(\eta), (i = 2, 4, 6, 8).$

In equations (16)-(23) the spectra  $\hat{V}_{z+}(w)$ ,  $\hat{I}_{z+}(w)$ ,  $\hat{V}_{\rho+}(w)$ ,  $\hat{I}_{\rho+}(w)$ ,  $\hat{V}_{z\pi+}(w_1)$ ,  $\hat{I}_{z\pi+}(w_1)$ ,  $\hat{V}_{\rho\pi+}(w_1)$ ,  $\hat{I}_{\rho\pi+}(w_1)$  will be called axial spectra and are defined by:

$$\begin{aligned}\hat{V}_{z+}(w) &= V_{z+}(-\tau_o \cos w, 0), \quad \hat{I}_{z+}(w) = I_{z+}(-\tau_o \cos w, 0), \quad \hat{V}_{\rho+}(w) = V_{\rho+}(-\tau_o \cos w, 0), \\ \hat{I}_{\rho+}(w) &= I_{\rho+}(-\tau_o \cos w, 0), \quad \hat{V}_{z\pi+}(w_1) = V_{z+}(-\tau_1 \cos w_1, \pi), \quad \hat{I}_{z\pi+}(w_1) = I_{z+}(-\tau_1 \cos w_1, \pi), \\ \hat{V}_{\rho\pi+}(w_1) &= V_{\rho+}(-\tau_1 \cos w_1, \pi), \quad \hat{I}_{\rho\pi+}(w_1) = I_{\rho+}(-\tau_1 \cos w_1, \pi).\end{aligned}$$

Equations (16)-(23) reduce the solution of our problem to the evaluation of the spectra  $\hat{Y}_i(w)$ ,  $\hat{Y}_i(w_1)$ . These functions will be obtained from the Fredholm factorization of the generalized equations (13),(14). The relevant FIE will be described in the next section.

### 3. The evaluation of the unknowns $\hat{Y}_i$ .

The author was able to obtain a closed form solution of the equations (13),(14) only for very special cases. Consequently, to obtain a general solution we must resort to semi-analytical factorization techniques. In particular the reduction of the GWHE to Fredholm integral equations of second kind (FIE) (Daniele,2004a) is very powerful. At normal incidence, the Fredholm factorization was successfully obtained in earlier papers (Daniele,2009,2010,2011; Daniele&Lombardi,2011). The presence of skew incidence requires some modifications reported in (Daniele, 2009). For the sake of brevity in this paper we summarize this procedure and in this section only we report the final form of the FIE (systems (24) and (25)).

$$\left\{ \begin{aligned} P_1(u) - \tan \frac{-\frac{\pi}{2} + ju}{2} P_2(u) - \int_{-\infty}^{\infty} M(u, u') P_2(u') du' &= n_1(u) \\ P_7(u) - \tan \frac{-\frac{\pi}{2} + ju}{2} P_8(u) - \int_{-\infty}^{\infty} M(u, u') P_8(u') du' &= n_7(u) \\ h_{ae11}(u) P_1(u) + \tan \frac{-\frac{\pi}{2} + jv(u)}{2} h_{ae33}(u) P_2(u) + \\ + \int_{-\infty}^{\infty} M_{ae32}(u, u') P_7(u') du' + \int_{-\infty}^{\infty} M_{ae33}(u, u') P_7(u') du' &= 0 \\ h_{ae22}(u) P_7(u) + \tan \frac{-\frac{\pi}{2} + jv(u)}{2} h_{ae44}(u) P_8(u) + \\ + \int_{-\infty}^{\infty} M_{ae41}(u, u') P_1(u') du' + \int_{-\infty}^{\infty} M_{ae44}(u, u') P_8(u') du' &= 0 \end{aligned} \right. \quad (24)$$

$$\left\{ \begin{array}{l}
P_3(u) - \tan \frac{-\frac{\pi}{2} + ju}{2} P_4(u) - \int_{-\infty}^{\infty} M(u, u') P_4(u') du' = n_3(u) \\
P_5(u) - \tan \frac{-\frac{\pi}{2} + ju}{2} P_6(u) - \int_{-\infty}^{\infty} M(u, u') P_6(u') du' = n_5(u) \\
h_{be11}(u) P_3(u) + \tan \frac{-\frac{\pi}{2} + jv(u)}{2} h_{be33}(u) P_4(u) + \\
+ \int_{-\infty}^{\infty} M_{be32}(u, u') P_5(u') du' + \int_{-\infty}^{\infty} M_{be33}(u, u') P_4(u') du' = 0 \\
h_{be22}(u) P_5(u) + \tan \frac{-\frac{\pi}{2} + jv(u)}{2} h_{be44}(u) P_6(u) + \\
+ \int_{-\infty}^{\infty} M_{be41}(u, u') P_3(u') du' + \int_{-\infty}^{\infty} M_{be44}(u, u') P_6(u') du' = 0
\end{array} \right. \quad (25)$$

In the two uncoupled FIE systems of order four above, eight new unknowns  $P_i(u)$   $i = 1, \dots, 8$  have been introduced. They provide the values of  $\hat{Y}_j$  through the equations (26)-(27). The known functions  $v(u)$ ,  $M(u, u')$ ,  $h_{aers}(u)$ ,  $h_{bers}(u)$ ,  $M_{aers}(u, u')$ ,  $M_{bers}(u, u')$ ,  $n_r(u)$  are defined in Appendix B. In particular the functions  $M(u, u')$ ,  $M_{aers}(u, u')$  and  $M_{bers}(u, u')$  are the kernels of the FIE and the functions  $n_r(u)$  are representative of source terms.

$$\hat{Y}_i(w) = \frac{1}{\tau_o} \left[ \int_{-\infty}^{\infty} M(-j(\frac{\pi}{\Phi} w + \frac{\pi}{2}), u') P_{i+1}(u') du' + n_i(w) \right] \quad (i = 1, 3, 5, 7) \quad (26)$$

$$\hat{Y}_{i+1}(w_1) = \frac{1}{\tau_o} \int_{-\infty}^{\infty} M(-j(\frac{\pi}{\Phi_1} w_1 + \frac{\pi}{2}), v') Q_{i+1}(v') dv' \quad (i = 1, 3, 5, 7) \quad (27)$$

In equation (27) the functions  $Q_i(v)$  are related to the  $P_i(u)$  through the representations:

$$\begin{pmatrix} Q_1(v) \\ Q_7(v) \\ Q_2(v) \\ Q_8(v) \end{pmatrix} = \int_{-\infty}^{\infty} T_{ae}(v, u) \cdot \begin{pmatrix} P_1(u) \\ P_7(u) \\ P_2(u) \\ P_8(u) \end{pmatrix} du, \quad \begin{pmatrix} Q_3(v) \\ Q_5(v) \\ Q_4(v) \\ Q_6(v) \end{pmatrix} = \int_{-\infty}^{\infty} T_{be}(v, u) \cdot \begin{pmatrix} P_3(u) \\ P_5(u) \\ P_4(u) \\ P_6(u) \end{pmatrix} du \quad (28)$$

where the known matrices  $T_{ae}(v, u)$  and  $T_{be}(v, u)$  of order four are defined in Appendix B.

#### 4. Analytical continuation of the starting axial spectra



The kernels  $M(u, u')$ ,  $M_{aers}(u, u')$ ,  $M_{bers}(u, u')$  are compact operators (Daniele, 2004b, 2009). Consequently, as has been shown by dozens of numerical simulations (Daniele 2009, 2011; Daniele & Lombardi, 2011), the solution of the FIE is very efficient and requires a very short execution time. However the solutions of (24) and (25) provide accurate representations of the analytical functions defined by (26) and (27) only in the strips  $-\Phi \leq \text{Re}[w] \leq \Phi$  and  $-\Phi_1 \leq \text{Re}[w_1] \leq \Phi_1$  respectively (Daniele, 2010). Since these analytical elements are not sufficient to get the axial spectra (equations (16)-(23)), a process of analytical continuation is necessary. For the normal incidence the analytical continuation has been considered in Daniele (2011). In this section we extend our procedure at the skew incidence. To accomplish it, firstly we must eliminate the unknowns  $X_{i+}$  ( $i = 1, \dots, 8$ ) in the equations (3)-(10). This task is considerable simplified if we rewrite the GWHE in the  $w$  and  $w_1$  planes defined through equations (15). In particular, in the plane  $w$  equations (3)-(6) assume the form (Daniele, 2009):

$$\begin{aligned} -Z_o \cos \beta \cos w \hat{I}_{z+}(w) + Z_o \sin \beta \hat{I}_{\rho+}(w) + \sin w \hat{V}_{z+}(w) = \\ -Z_o \cos \beta \cos(w + \Phi) \hat{I}_{az+}(w + \Phi) + Z_o \sin \beta \hat{I}_{a\rho+}(w + \Phi) + \sin(w + \Phi) \hat{V}_{az+}(w + \Phi) \end{aligned} \quad (29)$$

$$\begin{aligned} Z_o \sin w \hat{I}_{z+}(w) + \cos \beta \cos w \hat{V}_{z+}(w) - \sin \beta \hat{V}_{\rho+}(w) = \\ Z_o \sin(w + \Phi) \hat{I}_{az+}(w + \Phi) + \cos \beta \cos(w + \Phi) \hat{V}_{az+}(w + \Phi) - \sin \beta \hat{V}_{a\rho+}(w + \Phi) \end{aligned} \quad (30)$$

$$\begin{aligned} -Z_o \cos \beta \cos w \hat{I}_{z+}(w) + Z_o \sin \beta \hat{I}_{\rho+}(w) - \sin w \hat{V}_{z+}(w) = \\ -Z_o \cos \beta \cos(w + \Phi) \hat{I}_{bz+}(w + \Phi) + Z_o \sin \beta \hat{I}_{b\rho+}(w + \Phi) - \sin(w + \Phi) \hat{V}_{bz+}(w + \Phi) \end{aligned} \quad (31)$$

$$\begin{aligned} Z_o \sin w \hat{I}_{z+}(w) - \cos \beta \cos w \hat{V}_{z+}(w) + \sin \beta \hat{V}_{\rho+}(w) = \\ Z_o \sin(w + \Phi) \hat{I}_{bz+}(w + \Phi) - \cos \beta \cos(w + \Phi) \hat{V}_{bz+}(w + \Phi) + \sin \beta \hat{V}_{b\rho+}(w + \Phi) \end{aligned} \quad (32)$$

where the facial spectra in the  $w$ -plane are defined by:

$$\begin{aligned} \hat{V}_{az,bz+}(w) = V_{z+}(-\tau_o \cos(w), \pm\Phi) \quad , \quad \hat{I}_{a\rho,b\rho+}(w) = I_{\rho+}(-\tau_o \cos(w), \pm\Phi) \\ \hat{V}_{a\rho,b\rho+}(w) = V_{\rho+}(-\tau_o \cos(w), \pm\Phi) \quad , \quad \hat{I}_{az,bz+}(w) = I_{z+}(-\tau_o \cos(w), \pm\Phi) \end{aligned}$$

Similarly in the plane  $w_1$  equations (7)-(10) assume the form:

$$\begin{aligned}
& -Z_1 \cos \beta_1 \cos w_1 \hat{I}_{z\pi+}(w_1) + Z_1 \sin \beta_1 \hat{I}_{\rho\pi+}(w_1) + \sin w_1 \hat{V}_{z\pi+}(w_1) = \\
& = -Z_1 \cos \beta_1 \cos(w_1 + \Phi_1) \hat{I}_{bz+}(w_1 + \Phi_1) + Z_1 \sin \beta_1 \hat{I}_{b\rho+}(w_1 + \Phi_1) + \\
& + \sin(w_1 + \Phi_1) \hat{V}_{bz+}(w_1 + \Phi_1)
\end{aligned} \tag{33}$$

$$\begin{aligned}
& Z_1 \sin w_1 \hat{I}_{z\pi+}(w_1) + \cos \beta_1 \cos w_1 \hat{V}_{z\pi+}(w_1) - \sin \beta_1 \hat{V}_{\rho\pi+}(w_1) = \\
& = Z_1 \sin(w_1 + \Phi_1) \hat{I}_{bz+}(w_1 + \Phi_1) + \\
& + \cos \beta_1 \cos(w_1 + \Phi_1) \hat{V}_{bz+}(w_1 + \Phi_1) - \sin \beta_1 \hat{V}_{b\rho+}(w_1 + \Phi_1)
\end{aligned} \tag{34}$$

$$\begin{aligned}
& -Z_1 \cos \beta_1 \cos w_1 \hat{I}_{z\pi+}(w_1) + Z_1 \sin \beta_1 \hat{I}_{\rho\pi+}(w_1) - \sin w_1 \hat{V}_{z\pi+}(w_1) = \\
& = -Z_1 \cos \beta_1 \cos(w_1 + \Phi_1) \hat{I}_{az+}(w_1 + \Phi_1) + Z_1 \sin \beta_1 \hat{I}_{a\rho+}(w_1 + \Phi_1) + \\
& - \sin(w_1 + \Phi_1) \hat{V}_{az+}(w_1 + \Phi_1)
\end{aligned} \tag{35}$$

$$\begin{aligned}
& Z_1 \sin w_1 \hat{I}_{z\pi+}(w_1) - \cos \beta_1 \cos w_1 \hat{V}_{z\pi+}(w_1) + \sin \beta_1 \hat{V}_{\rho\pi+}(w_1) = \\
& = Z_1 \sin(w_1 + \Phi_1) \hat{I}_{az+}(w_1 + \Phi_1) - \cos \beta_1 \cos(w_1 + \Phi_1) \hat{V}_{az+}(w_1 + \Phi_1) + \\
& + \sin \beta_1 \hat{V}_{a\rho+}(w_1 + \Phi_1)
\end{aligned} \tag{36}$$

where the facial spectra in the  $w_1$ -plane are defined by:

$$\begin{aligned}
\hat{V}_{az,bz+}(w_1) &= V_{z+}(-\tau_1 \cos(w_1), \pm\Phi) \quad , \quad \hat{I}_{a\rho,b\rho+}(w_1) = I_{\rho+}(-\tau_1 \cos(w_1), \pm\Phi) \\
\hat{V}_{a\rho,b\rho+}(w_1) &= V_{\rho+}(-\tau_1 \cos(w_1), \pm\Phi) \quad , \quad \hat{I}_{az,bz+}(w_1) = I_{v+}(-\tau_1 \cos(w_1), \pm\Phi)
\end{aligned}$$

From (15) we have

$$\begin{aligned}
\hat{V}_{a\rho+,b\rho+}(w_1) &= \hat{V}_{a\rho+,b\rho+}(w), \quad \hat{I}_{a\rho+,b\rho+}(w_1) = \hat{I}_{a\rho+,b\rho+}(w) \\
\hat{V}_{az+,bz+}(w_1) &= \hat{V}_{az+,bz+}(w), \quad \hat{I}_{az+,bz+}(w_1) = \hat{I}_{az+,bz+}(w)
\end{aligned} \tag{37}$$

Besides  $w_1$  and  $w$  are related through the equations:

$$w_1 = g_t(w), \quad w = g_{t1}(w_1) \tag{38}$$

where:

$$g_t(w) = -\arccos\left(\frac{\cos w}{\lambda_e}\right), \quad g_{t1}(w_1) = -\arccos(\lambda_e \cos w_1)$$

To take advantage of equations (37), next we substitute  $w$  with  $\pm w - \Phi$  in equations (29)-(32) and  $w_1$  with  $\pm w_1 - \Phi_1$  in equations (33)-(36). In this way we get sixteen independent equations that contain the sixteen facial spectra  $\hat{V}_{az,bz+}(\pm w)$ ,  $\hat{I}_{az,bz+}(\pm w)$ ,  $\hat{V}_{a\rho,b\rho+}(\pm w)$ ,  $\hat{I}_{a\rho,b\rho+}(\pm w)$ . The other unknowns are the axial spectra. All the plus functions are always even functions (Daniele, 2003a, 2009, 2011), consequently we have eight additional equations  $\hat{V}_{az,bz+}(w) = \hat{V}_{az,bz+}(-w)$ ,  $\hat{I}_{az,bz+}(w) = \hat{I}_{az,bz+}(-w)$ ,  $\hat{V}_{a\rho,b\rho+}(w) = \hat{V}_{a\rho,b\rho+}(-w)$ ,  $\hat{I}_{a\rho,b\rho+}(w) = \hat{I}_{a\rho,b\rho+}(-w)$ . By eliminating the sixteen unknowns  $\hat{V}_{az,bz+}(\pm w)$ ,  $\hat{I}_{az,bz+}(\pm w)$ ,  $\hat{V}_{a\rho,b\rho+}(\pm w)$ ,  $\hat{I}_{a\rho,b\rho+}(\pm w)$  in the total of the twenty four equations above, we have at our disposal eight independent recursive equations that relate the axial spectra  $\hat{V}_{z+}(w)$ ,  $\hat{I}_{z+}(w)$ ,  $\hat{V}_{z\pi+}(w_1)$ ,  $\hat{I}_{z\pi+}(w_1)$ ,  $\hat{V}_{\rho+}(w)$ ,  $\hat{I}_{\rho+}(w)$ ,  $\hat{V}_{\rho\pi+}(w_1)$ ,  $\hat{I}_{\rho\pi+}(w_1)$ . After tedious algebraic manipulations (Daniele, 2009) these equations can be rewritten as:

$$\begin{aligned} \begin{vmatrix} \hat{V}_{\rho+}(w) \\ \hat{V}_{z+}(w) \\ \hat{I}_{z+}(w) \\ \hat{I}_{\rho+}(w) \end{vmatrix} &= H_{11n}[w + \Phi, g_t(w + \Phi)] \begin{vmatrix} \hat{V}_{\rho+}(w + 2\Phi) \\ \hat{V}_{z+}(w + 2\Phi) \\ \hat{I}_{z+}(w + 2\Phi) \\ \hat{I}_{\rho+}(w + 2\Phi) \end{vmatrix} + \\ &+ H_{12n}[w + \Phi, g_t(w + \Phi)] \begin{vmatrix} \hat{V}_{\rho\pi+}(g_t(w + \Phi) + \Phi_1) \\ \hat{V}_{z\pi+}(g_t(w + \Phi) + \Phi_1) \\ \hat{I}_{z\pi+}(g_t(w + \Phi) + \Phi_1) \\ \hat{I}_{\rho\pi+}(g_t(w + \Phi) + \Phi_1) \end{vmatrix} \end{aligned} \quad (39)$$

$$\begin{aligned} \begin{vmatrix} \hat{V}_{\rho\pi+}(w_1) \\ \hat{V}_{z\pi+}(w_1) \\ \hat{I}_{z\pi+}(w_1) \\ \hat{I}_{\rho\pi+}(w_1) \end{vmatrix} &= H_{21n}[g_{t1}(w_1 + \Phi_1), w_1 + \Phi_1] \begin{vmatrix} \hat{V}_{\rho+}(g_{t1}(w_1 + \Phi_1) + \Phi) \\ \hat{V}_{z+}(g_{t1}(w_1 + \Phi_1) + \Phi) \\ \hat{I}_{z+}(g_{t1}(w_1 + \Phi_1) + \Phi) \\ \hat{I}_{\rho+}(g_{t1}(w_1 + \Phi_1) + \Phi) \end{vmatrix} + \\ &+ H_{22n}[g_{t1}(w_1 + \Phi_1), w_1 + \Phi_1] \begin{vmatrix} \hat{V}_{\rho\pi+}(w_1 + 2\Phi_1) \\ \hat{V}_{z\pi+}(w_1 + 2\Phi_1) \\ \hat{I}_{z\pi+}(w_1 + 2\Phi_1) \\ \hat{I}_{\rho\pi+}(w_1 + 2\Phi_1) \end{vmatrix} \end{aligned} \quad (40)$$

where the known matrices  $H_{11n}(w, w_1)$ ,  $H_{12n}(w, w_1)$ ,  $H_{21n}(w, w_1)$ ,  $H_{22n}(w, w_1)$  have very cumbersome expressions that are reported in Daniele (2009).

The presence of the complicate functions  $g_t(w)$  and  $g_{t1}(w_1)$  means that great care is required to deal with the above equations. However, it has been proved that suitable iterations of (39) and (40) allow the evaluation of the axial spectra for every value of  $w$  and  $w_1$  (Daniele 2009; Daniele&Lombardi,2011).

## 5. Spectra for every direction $\varphi$

Given the whole axial spectra we can obtain the spectra for every direction  $\varphi$  by using the rotating waves (2003b). In the free space we get the following expressions of the spectra for every direction  $-\Phi \leq \varphi \leq \Phi$  :

$$V_{z+}(-\tau_o \cos w, \varphi) = \frac{v_1(w + \varphi) + v_2(w - \varphi)}{-\tau_o \sin w}, \quad I_{z+}(-\tau_o \cos w, \varphi) = \frac{i_1(w + \varphi) + i_2(w - \varphi)}{-\tau_o \sin w}$$

where the rotating waves  $v_1(w)$ ,  $i_1(w)$ ,  $v_2(w) = -v_1(-w)$  and  $i_2(w) = -i_1(-w)$  are obtained in terms of the axial spectra  $\hat{V}_{z+}(w)$ ,  $\hat{I}_{z+}(w)$  by the following equations:

$$v_1(w) = -\frac{1}{2}k_o \sin \beta (Z_o \cos w \cos \beta \hat{I}_{z+}(w) - Z_o \sin \beta \hat{I}_{\rho+}(w) + \sin(w) \hat{V}_{z+}(w))$$

$$i_1(w) = -\frac{k_o \sin \beta (Z_o \sin w \hat{I}_{z+}(w) - \cos \beta \cos w \hat{V}_{z+}(w) + \sin(\beta) \hat{V}_{\rho+}(w))}{2Z_o}$$

Similarly in the wedge ( $-\Phi_1 \leq \varphi_1 \leq \Phi_1$ ,  $\varphi_1 = \pi - \varphi$ ) we get:

$$V_{z+}(-\tau_1 \cos w_1, \varphi) = \frac{v_{1d}(w_1 + \varphi) + v_{2d}(w_1 - \varphi)}{-\tau_1 \sin w_1}, \quad I_{z+}(-\tau_1 \cos w_1, \varphi) = \frac{i_{1d}(w_1 + \varphi) + i_{2d}(w_1 - \varphi)}{-\tau_1 \sin w_1}$$

where the rotating waves are expressed in terms of the axial spectra  $\hat{V}_{z\pi+}(w)$ ,  $\hat{I}_{z\pi+}(w)$  by the following equations:

$$v_{d1}(w_1) = -v_{d2}(-w_1) =$$

$$= -\frac{1}{2}k_1 \sin \beta_1 (-Z_1 \cos w_1 \cos \beta_1 \hat{I}_{z\pi+}(-\pi + w_1) - Z_1 \sin \beta_1 \hat{I}_{\rho\pi+}(-\pi + w_1) - \sin(w_1) \hat{V}_{z\pi+}(-\pi + w_1))$$

$$i_{d1}(w_1) = -i_{d1}(-w_1) =$$

$$= -\frac{k_1 \sin \beta_1 (-Z_1 \sin w_1 \hat{I}_{z\pi+}(-\pi + w_1) + \cos(w_1) \cos \beta_1 \hat{V}_{z\pi+}(-\pi + w_1) + \sin \beta_1 \hat{V}_{\rho\pi+}(-\pi + w_1))}{2Z_1}$$

## 6. Conclusion

This paper describes the solution of the diffraction problem indicated in fig.1 where a plane wave impinges on a penetrable wedge at skew incidence angle  $\beta$ . This solution is based on the W-H technique and generalizes the solution obtained in Daniele (2010,11) at the normal incidence. In the skew incidence case, even though the number of the unknowns doubles, no new conceptual and numerical difficulties are present. However, the procedure used becomes very heavy. For the sake of brevity, sometimes it has only been outlined. The Daniele Report (2009) illustrates several details of this work. It is freely available for download. By using the MATHEMATICA computer program

(Daniele ,2009), dozens of diffraction diagrams have been plotted in very short execution time. They are not reported here.

For  $\beta = \pi/2$  (normal incidence),  $\mu_r = 1$ ,  $H_o = 0$ , the validation of the solution obtained by the W-H technique has been ascertained in several papers (Daniele 2010,2011,Daniele&Lombardi 2011). In particular in Daniele&Lombardi (2011) four tests cases were studied in detail. For the skew incidence case new test cases will be considered in a future paper.

## Appendix A Some details on the Fredholm integral equations

Equation (13) can be reduced to a classical WH equation (A2) by introducing the mapping (Daniele 2001,2003a) :

$$\eta = -\tau_o \cos \left[ \frac{\Phi}{\pi} \arccos \left( -\frac{\alpha}{\tau} \right) \right] \quad (A1)$$

$$\bar{Y}_{(i)+}(\alpha) = \bar{X}_{(i)-}(\alpha) + \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \bar{X}_{(i+1)-}(\alpha) \quad (i = 1, 3, 5, 7) \quad (A2)$$

where  $\bar{Y}_{(i)+}(\alpha) = Y_{(i)+}(\eta)$ ,  $\bar{X}_{(i)-}(\alpha) = X_{(i)+}(-m)$

The similar equation (14) becomes the classical WH equation (A3) by introducing the

$$\text{mapping } \eta = -\tau_1 \cos \left[ \frac{\Phi_1}{\pi} \arccos \left( -\frac{\alpha_1}{\tau_1} \right) \right]$$

$$\bar{Y}_{(i+1)+}(\alpha_1) = \ddot{X}_{(i)-}(\alpha_1) - \frac{\tau_1 + \alpha_1}{\sqrt{\tau_1^2 - \alpha_1^2}} \ddot{X}_{(i+1)-}(\alpha_1) \quad (i = 1, 3, 5, 7) \quad (A3)$$

where:  $\bar{Y}_{(i+1)+}(\alpha_1) = Y_{(i+1)+}(\eta)$ ,  $\ddot{X}_{(i)-}(\alpha_1) = \dot{X}_{(i)+}(-m_1)$

Since there are not sources in the interior of the dielectric wedge, the standard procedure for reducing WHE to FIE (Daniele,2004a, 2005) yields the FIE equation (A4) without any difficulty:

$$\begin{aligned} & \ddot{X}_{(i)-}(\alpha_1) - \frac{\tau_1 + \alpha_1}{\sqrt{\tau_1^2 - \alpha_1^2}} \ddot{X}_{(i+1)-}(\alpha_1) + \\ & + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left( \frac{\tau_1 + \alpha_1'}{\sqrt{\tau_1^2 - \alpha_1'^2}} - \frac{\tau_1 + \alpha_1}{\sqrt{\tau_1^2 - \alpha_1^2}} \right) \ddot{X}_{(i+1)-}(\alpha_1') \frac{d\alpha_1'}{\alpha_1' - \alpha_1} = 0 \end{aligned} \quad (i = 1, 3, 5, 7) \quad (A4)$$

Conversely, the equation (A2) requires the source contribution to be taken into account. It yields (Daniele,2009) the not homogeneous FIE (A5):

$$\begin{aligned} & \bar{X}_{(i)-}(\alpha) + \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \bar{X}_{(i+1)-}(\alpha) + \\ & - \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\left( \frac{\tau_o + \alpha'}{\sqrt{\tau_o^2 - \alpha'^2}} - \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \right) \bar{X}_{(i+1)-}(\alpha')}{\alpha' - \alpha} d\alpha' = \bar{n}_i(\alpha), \quad (i = 1, 3, 5, 7) \end{aligned} \quad (\text{A5})$$

The two uncoupled FIE systems (24) and (25) derive from equations (A4) and (A5) through algebraic manipulations. In order to obtain this deduction, a very difficult step is to relate the minus functions  $\ddot{X}_{(i)-}(\alpha_1)$  and  $\bar{X}_{(i)-}(\alpha)$  because they are defined in the two different complex planes  $\alpha$  and  $\alpha_1$ . For the sake of brevity we omit the details of the procedure we used. They are reported in Daniele (2005, 2010) for the normal case and in Daniele (2009) for the general skew case.

The explicit expression of the source term  $\bar{n}_i(\alpha)$  (equation) (A6) requires several steps that for the skew incidence case are reported in (Daniele, 2009). We get:

$$\begin{aligned} \bar{n}_1(\alpha) &= \text{If}[\varphi_o < \frac{\Phi}{2}, \frac{-j4E_o \frac{\pi}{\Phi}}{(\alpha - \alpha_o)}, \frac{R_{10}}{(\alpha - \alpha_o)} + \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \frac{R_{20}}{(\alpha - \alpha_o)}] \\ \bar{n}_3(\alpha) &= \text{If}[\varphi_o < \frac{\Phi}{2}, \frac{j4E_o \frac{\pi}{\Phi} \sin(\frac{\pi \varphi_o}{2\Phi})}{\alpha - \alpha_o}, \frac{R_{30}}{(\alpha - \alpha_o)} + \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \frac{R_{40}}{(\alpha - \alpha_o)}] \\ \bar{n}_5(\alpha) &= \text{If}[\varphi_o < \frac{\Phi}{2}, \frac{j4H_o \frac{\pi}{\Phi}}{\alpha - \alpha_o}, \frac{R_{50}}{(\alpha - \alpha_o)} + \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \frac{R_{60}}{(\alpha - \alpha_o)}] \\ \bar{n}_7(\alpha) &= \text{If}[\varphi_o < \frac{\Phi}{2}, \frac{-j4H_o \frac{\pi}{\Phi} \sin(\frac{\pi \varphi_o}{2\Phi})}{\alpha - \alpha_o}, \frac{R_{70}}{(\alpha - \alpha_o)} + \frac{\tau_o + \alpha}{\sqrt{\tau_o^2 - \alpha^2}} \frac{R_{80}}{(\alpha - \alpha_o)}] \end{aligned} \quad (\text{A6})$$

In equations (A6) the MATHEMATICA function  $\text{If}[c, a, b]$  gives  $a$  if the condition  $c$  evaluates to True, and  $b$  if the condition  $c$  evaluates to False.

The expressions of the coefficients  $R_{i0}$  are given by (Daniele, 2009):

$$\begin{aligned} R_{10} &= -2j \frac{\pi}{\Phi} (E_o + E_{or}), \quad R_{20} = -2j \frac{\pi}{\Phi} \cot(\frac{\pi \varphi_o}{2\Phi}) (E_o - E_{or}) \\ R_{30} &= 2j \frac{\pi}{\Phi} (E_o + E_{or}) \sin(\frac{\pi \varphi_o}{2\Phi}), \quad R_{40} = 2j \frac{\pi}{\Phi} (E_o - E_{or}) \cos(\frac{\pi \varphi_o}{2\Phi}) \\ R_{50} &= 2j \frac{\pi}{\Phi} (H_o + H_{or}), \quad R_{60} = 2j \frac{\pi}{\Phi} \cot(\frac{\pi \varphi_o}{2\Phi}) (H_o - H_{or}) \\ R_{70} &= -2j \frac{\pi}{\Phi} (H_o + H_{or}) \sin(\frac{\pi \varphi_o}{2\Phi}), \quad R_{80} = -2j \frac{\pi}{\Phi} (H_o - H_{or}) \cos(\frac{\pi \varphi_o}{2\Phi}) \end{aligned} \quad (\text{A8})$$

By supposing  $\varphi_o > 0$ ,  $E_{or}$  and  $H_{or}$  are the intensity of the plane wave (A9) reflected by the face a:

$$E_z^r(\rho, \varphi) = E_{or} e^{j\tau_o \rho \cos(\varphi - \varphi_r)}, H_z^r(\rho, \varphi) = H_{or} e^{j\tau_o \rho \cos(\varphi - \varphi_r)} \quad (\text{A9})$$

with  $\varphi_r = 2\Phi - \varphi_o$

Similarly  $E_{ot}$  and  $H_{ot}$  are the intensity of the plane wave (A10) transmitted in the wedge by the face a:

$$E_z^t(\rho, \varphi) = E_{ot} e^{j\tau_1 \rho \cos(\varphi - \varphi_t)}, H_z^t(\rho, \varphi) = H_{ot} e^{j\tau_1 \rho \cos(\varphi - \varphi_t)} \quad (\text{A10})$$

with  $\varphi_t = \Phi + g_t(\Phi - \varphi_o)$  and  $g_t(w) = -\arccos\left(\frac{\cos w}{\lambda_e}\right)$ .

In order to get  $E_{or}, H_{or}, E_{ot}$  and  $H_{ot}$  we must take into account the equations (A11),(A12) that relates the transversal components to the longitudinal components :

$$E_\rho^{r,t}(\rho, \varphi) = \frac{k_{o,1}}{j\tau_{o,1}^2} \left( \frac{\alpha_{o,1}}{k_{o,1}} \frac{\partial E_z^{r,t}(\rho, \varphi)}{\partial \rho} + \frac{Z_{o,1}}{\rho} \frac{\partial H_z^{r,t}(\rho, \varphi)}{\partial \varphi} \right) \quad (\text{A11})$$

$$H_\rho^{r,t}(\rho, \varphi) = \frac{k_{o,1}}{j\tau_{o,1}^2} \left( \frac{\alpha_{o,1}}{k_{o,1}} \frac{\partial H_z^{r,t}(\rho, \varphi)}{\partial \rho} - \frac{Y_{o,1}}{\rho} \frac{\partial E_z^{r,t}(\rho, \varphi)}{\partial \varphi} \right) \quad (\text{A12})$$

By forcing the four boundary conditions on the face a:

$$E_z^i(\rho, \Phi) + E_z^r(\rho, \Phi) = E_z^t(\rho, \Phi), \quad E_\rho^i(\rho, \Phi) + E_\rho^r(\rho, \Phi) = E_\rho^t(\rho, \Phi),$$

$$H_z^i(\rho, \Phi) + H_z^r(\rho, \Phi) = H_z^t(\rho, \Phi), \quad H_\rho^i(\rho, \Phi) + H_\rho^r(\rho, \Phi) = H_\rho^t(\rho, \Phi)$$

we get a system of four equations that provides the evaluation of  $E_{or}, H_{or}, E_{ot}$  and  $H_{ot}$ . The expressions (Daniele,2009) of  $E_{or}, H_{or}, E_{ot}$  and  $H_{ot}$  are very weighty and not reported in this paper.

## Appendix B.

In the sequel we report the known functions that define the parameters of the two systems (24) and (25):

$$\varepsilon_{tr} = \frac{\tau_1^2}{\tau_o^2} = \lambda_e^2 = 1 + (\varepsilon_r \mu_r - 1) \csc^2 \beta$$

$$v(u) = \frac{j\pi}{2\Phi_1} \left( \Phi_1 - 2 \arccos \frac{\cos \frac{(\pi + 2ju)\Phi}{2\pi}}{\sqrt{\varepsilon_{tr}}} \right)$$

$$M(u, u') = \frac{1}{\pi j} \frac{e^u}{(e^u - j)} \frac{e^{u'} + j}{e^{u+u'} + 1}$$

$$h_{ae11}(u) = \cosh(u) \csc \frac{(\pi + 2ju)\Phi}{2\pi} \operatorname{sech}(v(u)) \sin \frac{(\pi + 2jv(u))\Phi}{2\pi}$$

$$h_{ae22}(u) = \cos \frac{1}{4}(\pi - 2ju) \csc \frac{(\pi + 2ju)\Phi}{2\pi} \sec \frac{1}{4}(\pi - 2jv(u)) \sin \frac{(\pi + 2jv(u))\Phi}{2\pi}$$

$$h_{ae32}(u) = Z_1 \cos \frac{1}{4}(\pi - 2ju) \cos(\beta) \cdot$$

$$[\varepsilon_{tr} \cot \frac{(\pi + 2ju)\Phi}{2\pi} - \cos \frac{(\pi + 2jv(u))\Phi}{2\pi} \csc \frac{(\pi + 2ju)\Phi}{2\pi}] \csc^2 \frac{(\pi - 2jv(u))}{4}$$

$$h_{ae33}(u) = \frac{Z_1 \varepsilon_{tr} (j + \sinh u)}{Z_o (j + \sinh v(u))}$$

$$h_{ae41}(u) = \cos(\beta) \cosh(u) \cdot$$

$$\frac{[\varepsilon_{tr} \cot \frac{(\pi + 2ju)\Phi}{2\pi} - \cos \frac{(\pi + 2jv(u))\Phi}{2\pi} \csc \frac{(\pi + 2ju)\Phi}{2\pi}] \csc \frac{(\pi - 2jv(u))}{4}}{2Z_o}$$

$$h_{ae44}(u) = \varepsilon_{tr} \csc \frac{(\pi - 2jv(u))}{4} \sin \frac{(\pi - 2ju)}{4}, \quad h_{be22}(u) = h_{ae11}(u), \quad h_{be11}(u) = -h_{ae22}(u)$$

$$h_{be32}(u) = Z_1 Z_o h_{ae41}(u), \quad h_{be33}(u) = \frac{Z_1}{Z_o} h_{ae44}(u), \quad h_{be41}(u) = -\frac{1}{Z_1 Z_o} h_{ae32}(u),$$

$$h_{be44}(u) = \frac{Z_o}{Z_1} h_{ae33}(u)$$

$$M_{aeij}(u, u') = \int_{-\infty}^{\infty} M(v(u), v') T_{aeij}(v', u') dv', \quad M_{beij}(u, u') = \int_{-\infty}^{\infty} M(v(u), v') T_{beij}(v', u') dv'$$

$$T(v, u) = -\frac{\Phi}{2\pi^2} \frac{\sin[\frac{\Phi}{\pi}(ju + \frac{\pi}{2}) + \Phi]}{\cos[\frac{\Phi}{\pi}(ju + \frac{\pi}{2}) + \Phi] - \sqrt{\varepsilon_{tr}} \cos[\frac{\Phi}{\pi}(jv + \frac{\pi}{2}) + \Phi_1]}$$



$$T_{ae}(v, u) = \begin{vmatrix} T_{ae11} & 0 & 0 & 0 \\ 0 & T_{ae22} & 0 & 0 \\ 0 & T_{ae32} & T_{ae33} & 0 \\ T_{ae41} & 0 & 0 & T_{ae44} \end{vmatrix}, \quad T_{be}(v, u) = \begin{vmatrix} T_{be11} & 0 & 0 & 0 \\ 0 & T_{be22} & 0 & 0 \\ 0 & T_{be32} & T_{be33} & 0 \\ T_{be41} & 0 & 0 & T_{be44} \end{vmatrix}$$

$$T_{ae11} = T(v, u) \cosh(u) \csc \frac{(\pi + 2ju)\Phi}{2\pi} \operatorname{sech}(v) \sin \frac{(\pi + 2jv)\Phi_1}{2\pi}$$

$$T_{ae22} = T(v, u) \cos \frac{1}{4}(\pi - 2ju) \csc \frac{(\pi + 2ju)\Phi}{2\pi} \sec \frac{1}{4}(\pi - 2jv) \sin \frac{(\pi + 2jv)\Phi_1}{2\pi}$$

$$T_{ae32} = T(v, u) Z_1 \cos \frac{1}{4}(\pi - 2ju) \cos(\beta)$$

$$[\varepsilon_{ir} \cot \frac{(\pi + 2ju)\Phi}{2\pi} - \cos \frac{(\pi + 2jv)\Phi_1}{2\pi} \csc \frac{(\pi + 2ju)\Phi}{2\pi}] \csc^2 \frac{(\pi - 2jv)}{4}$$

$$T_{ae33} = T(v, u) \frac{Z_1 \varepsilon_{ir} (j + \sinh u)}{Z_o (j + \sinh v)}$$

$$T_{ae41} = T(v, u) \cos(\beta) \cosh(u)$$

$$\frac{[\varepsilon_{ir} \cot \frac{(\pi + 2ju)\Phi}{2\pi} - \cos \frac{(\pi + 2jv)\Phi_1}{2\pi} \csc \frac{(\pi + 2ju)\Phi}{2\pi}] \csc \frac{(\pi - 2jv)}{4}}{2Z_o}$$

$$T_{ae44} = T(v, u) \varepsilon_{ir} \csc \frac{(\pi - 2jv)}{4} \sin \frac{(\pi - 2ju)}{4}, \quad T_{be22} = T_{ae11}, \quad T_{be11} = -T_{ae22}$$

$$T_{be32} = Z_1 Z_o T_{ae41}, \quad T_{be33} = \frac{Z_1}{Z_o} T_{ae44}, \quad T_{be41} = -\frac{1}{Z_1 Z_o} T_{ae32}, \quad T_{be44} = \frac{Z_o}{Z_1} T_{ae33}$$

$$\begin{aligned}
n_1(u) &= -If[\varphi_o < \frac{\Phi}{2}, \frac{-j4E_o \frac{\pi}{\Phi_i}}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}, \frac{R_{10} - R_{20} \tan \frac{ju - \pi/2}{2}}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}] \\
n_3(u) &= -If[\varphi_o < \frac{\Phi}{2}, \frac{j4E_o \frac{\pi}{\Phi} \sin(\frac{\pi \varphi_o}{2\Phi})}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}, \frac{R_{30} - R_{40} \tan \frac{ju - \pi/2}{2}}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}] \\
n_5(u) &= -If[\varphi_o < \frac{\Phi}{2}, \frac{j4H_o \frac{\pi}{\Phi}}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}, \frac{R_{50} - R_{60} \tan \frac{ju - \pi/2}{2}}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}] \\
n_7(u) &= -If[\varphi_o < \frac{\Phi}{2}, \frac{-j4H_o \frac{\pi}{\Phi} \sin(\frac{\pi \varphi_o}{2\Phi})}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}, \frac{R_{70} - R_{80} \tan \frac{ju - \pi/2}{2}}{j \sinh u - \cos(\frac{\pi}{\Phi} \varphi_o)}]
\end{aligned}$$

The explicit expressions of  $R_{io}$  as well as of the matrices  $H_{11n}(w, w_1)$ ,  $H_{12n}(w, w_1)$ ,  $H_{21n}(w, w_1)$ ,  $H_{22n}(w, w_1)$  are very cumbersome and reported in Appendix H of Daniele (2009).

### Appendix C. Characteristics of the kernels

For the sake of brevity this Appendix only refers to the case:  $\beta = \pi/2$ ,  $\mu_r = 1$ ,  $H_o = 0$ . In this case system (24) and (25) are of order two since the functions  $P_i(u)$  ( $i = 5, \dots, 8$ ) are vanishing. Moreover also the functions  $Q_i(u)$  ( $i = 5, \dots, 8$ ) are vanishing.

All the kernels present in the FIE (24) and (25) derive from the kernel

$$M(u, u') = \frac{1}{\pi j} \frac{e^u}{(e^u - j)} \frac{e^{u'} + j}{e^{u+u'} + 1}$$

A property of this kernel is that it satisfies the condition:

$$M(u, u') = M^*(-u, -u'). \quad (C1)$$

where  $*$  means complex conjugation.

Consequently by considering the complex conjugate of the equations (24) and (25) we obtain that  $-P_{1,2}^*(-u)$ ,  $-Q_{1,2}^*(-v)$  are solutions of the problem too.

Since the solution is unique it yields:

$$P_{1,2}(u) = -P_{1,2}^*(-u) \quad Q_{1,2}(u) = -Q_{1,2}^*(-u). \quad (C2)$$

The property (C2) has been ascertained numerically.

According to the general theory (Daniele,2004a), the kernels of the integral equations occurring in the Fredholm factorization are compact in a suitable functions space. In this Appendix this important property will be studied directly in the Hilbert space  $L_2$ . We first reduce the infinite interval of integration  $-\infty < u < \infty$  into a segment  $-1 \leq x \leq 1$ , through the well known transformation  $u = u(x) = \text{arctanh}(x)$  (do not confuse this new variable  $x$  with the geometrical coordinate  $x$  of Fig.1). After algebraic manipulations, in the  $x$ -domain ( $-1 \leq x \leq 1$ ), the integral equations (24) become:

$$\tilde{P}_1(x) - (j - (1 - x^2)^{1/2})\tilde{P}_2(x) - \frac{1}{\pi j} \int_{-1}^1 \frac{\Delta(x, x')}{(1 - x^2)^\delta (1 - x'^2)^{1/2-\delta}} \tilde{P}_2(x') dx' = N_2(x) \quad (C3)$$

$$C_1(x)\tilde{P}_1(x) + C_2(x)\tilde{P}_2(x) - \frac{1}{\pi j} \int_{-1}^1 \frac{B_{2r}(x, x')}{(1 - x^2)^\delta (1 - x'^2)^{1/2-\delta}} \tilde{P}_2(x') dx' = 0 \quad (C4)$$

where, being  $T(v, u)$  defined by (B3), we have:

$$\begin{aligned} \tilde{P}_i(x) &= \frac{P_i(u(x))}{(1 - x^2)^{1/2+\delta}}, i = 1, 2, \quad \Delta(x, x') = \frac{(x - x' + j((1 - x^2)^{1/2} - (1 - x'^2)^{1/2}))}{(1 - x^2)^{1/2} x' - (1 - x'^2)^{1/2} x} \\ N_2(x) &= \frac{4\pi E_o}{\Phi(1 - x^2)^\delta (x + j(1 - x^2)^{1/2} \cos \frac{\pi \varphi_o}{\Phi})}, \quad C_1(x) = \frac{\sin[\frac{jv(u(x))\Phi_1}{\pi} + \frac{\Phi_1}{2}]}{\sin[\frac{ju(x)\Phi}{\pi} + \frac{\Phi}{2}]} \\ C_2(x) &= \tan \frac{-\frac{\pi}{2} + jv(u(x))}{2} \frac{\cosh(v(u(x)))}{\cosh(u(x))} \frac{Z_1 \sin^2[\frac{1}{2}(-\frac{\pi}{2} + ju(x))]}{Z_o \sin^2[\frac{1}{2}(-\frac{\pi}{2} + jv(u(x)))]} \end{aligned}$$

$$\begin{aligned} B_{2r}(x, x') &= \int_{-\infty}^{\infty} B_2(v(u(x), v') R_2(u(x'), v') dv' \\ B_2(v, v') &= \frac{\cosh(v) e^v (e^{v'} + j)}{(e^v - j)(e^{v+v'} + 1) \sin^2[\frac{1}{2}(-\frac{\pi}{2} + jv')]} \\ R_2(u', v') &= -\frac{\Phi Z_1}{2\pi^2 Z_o} \sin^2[\frac{1}{2}(-\frac{\pi}{2} + ju')] T(v', u') \end{aligned}$$

In the equations (C3) and (C4), the real parameter  $\delta$  has been introduced for convenience. The range of this parameter has some limitations. For instance we must have  $\delta < 1/4$ , otherwise the vector  $N_2(x) \notin L_2$ . Taking into account the expressions of  $P_i(u)$ , we get:

$$\tilde{P}_i(x) = O\left[\frac{(1 - x^2)^{1/2 + \frac{\Phi}{2\pi}}}{(1 - x^2)^{1/2 + \delta}}\right], \quad i = 1, 2$$

Whence  $\delta < 1/4$  yields  $\tilde{P}_i(x) \in L_2$ ,  $i = 1, 2$ .

Taking into account that as  $u \rightarrow \pm\infty$  (or  $x = \pm 1$ ) :

$$v(u) \approx \frac{\Phi}{\Phi_1} u - \frac{\pi \log \varepsilon_r}{2\Phi_1} - j \frac{\pi(\Phi - \Phi_1)}{2\Phi_1} \quad (C5)$$

it follows that the matrix  $\begin{vmatrix} 1 & -(j - (1 - x^2)^{1/2}) \\ C_1(x) & C_2(x) \end{vmatrix}$  is invertible for every values of  $-1 \leq x \leq +1$ . For instance at  $x = \pm 1$ , it becomes  $\begin{vmatrix} 1 & -j \\ 1 & j \frac{Z_1}{Z_o} \end{vmatrix}$ .

Consequently to verify that the system (C3-C4) is Fredholm of second kind, we must study the two functions  $\Delta(x, x')$  and  $B_{2r}(x, x')$ . The function  $\Delta(x, x')$  is not bounded in the square  $-1 \leq x, x' \leq 1$ . However numerical simulations have ascertained that, for non vanishing positive values of the parameter  $\delta_1$ ,  $\text{Max} \{(1 - x'^2)^{\delta_1} \Delta(x, x')\} < \infty$  in the square  $-1 \leq x, x' \leq 1$  results. Whence we rewrite the kernel of (C3) in the form:

$$\frac{(1 - x'^2)^{\delta_1} \Delta(x, x')}{(1 - x^2)^{\delta + \delta_1} (1 - x'^2)^{1/2 - \delta + \delta_1}} \quad (C6)$$

Kernel (C6) is compact in  $L_2$  if  $\int_{-1}^1 \int_{-1}^1 \left| \frac{(1 - x'^2)^{\delta_1} \Delta(x, x')}{(1 - x^2)^{\delta + \delta_1} (1 - x'^2)^{1/2 - \delta + \delta_1}} \right|^2 dx dx' < \infty$ . Taking into account that the max of  $\left| (1 - x'^2)^{\delta_1} \Delta(x, x') \right|^2$  is finite in the square  $-1 \leq x, x' \leq 1$ , we easily verify this equation provided that  $0 < \delta_1 < \delta < \frac{1}{4}$ . Within the same conditions, also

the kernel  $\frac{B_{2r}(x, x')}{(1 - x^2)^{\delta} (1 - x'^2)^{1/2 - \delta}}$  is compact provided that

$$\left| B_{2r}(x, x') \right|^2 = \left| \int_{-\infty}^{\infty} B_2(v(u(x), v')) R_2(u(x'), v') dv' \right|^2 < \infty \quad -1 \leq x, x' \leq 1 \quad (C7)$$

To ascertain (C7) we observe that  $\text{Max} \{B_2(v, v')\} < \infty$  for every value of  $-\infty \leq v, v' \leq \infty$ . It occurs in the worst case  $v = -v'$  too. Furthermore numerical simulations put in evidence that  $\int_{-\infty}^{\infty} R_2(u(x'), v') dv'$  is finite for every  $-1 \leq x' \leq 1$ .

Whence, taking into account (C5), equation (C7) holds.

To ascertain the compactness of the kernels involved in system (25) requires a slight modification of the above proof. In particular the new system is :

$$\tilde{P}_3(x) - (j - (1 - x^2)^{1/2}) \tilde{P}_4(x) - \frac{1}{\pi j} \int_{-1}^1 \frac{\Delta(x, x')}{(1 - x^2)^{\delta} (1 - x'^2)^{1/2 - \delta}} \tilde{P}_4(x') dx' = N_4(x) \quad (C8)$$

$$C_1(x)\tilde{P}_3(x) + C_4(x)\tilde{P}_4(x) + \frac{1}{\pi j} \frac{2(-1)^{1/4}}{\sqrt{1-x} + j\sqrt{1+x}} \int_{-1}^1 \frac{B_{4r}(x, x')}{(1-x^2)^{1/4+\delta} (1-x'^2)^{1/2-\delta}} \tilde{P}_4(u') dx' = 0 \quad (C9)$$

where we have:

$$\tilde{P}_i(x) = \frac{P_i(u(x))}{(1-x^2)^{1/2+\delta}} \in L_2, \quad i = 3, 4, \quad \Delta(x, x') = \frac{(x-x' + j((1-x^2)^{1/2} - (1-x'^2)^{1/2}))}{(1-x^2)^{1/2} x' - (1-x'^2)^{1/2} x}$$

$$N_4(x) = \frac{4\pi E_o \sin(\frac{\pi\varphi_o}{2\Phi})}{\Phi(1-x^2)^\delta (x + j(1-x^2)^{1/2} \cos \frac{\pi\varphi_o}{\Phi})},$$

$$C_4(x) = \tan \frac{-\frac{\pi}{2} + jv(u(x)) \cos[\frac{1}{2}(-\frac{\pi}{2} + jv(u(x)))]}{2} \frac{Z_1 \sin[\frac{1}{2}(-\frac{\pi}{2} + ju(x))]}{\cos[\frac{1}{2}(-\frac{\pi}{2} + ju(x))]} \frac{Z_o \sin[\frac{1}{2}(-\frac{\pi}{2} + jv(u(x)))]}{Z_o \sin[\frac{1}{2}(-\frac{\pi}{2} + jv(u(x)))]}$$

$$B_{4r}(x, x') = \int_{-\infty}^{\infty} B_4(v(u(x), v') R_4(u(x'), v') dv'$$

$$B_4(v, v') = \frac{\cos[\frac{1}{2}(-\frac{\pi}{2} + jv)] e^v (e^{v'} + j)}{(e^v - j)(e^{v+v'} + 1) \sin[\frac{1}{2}(-\frac{\pi}{2} + jv')]}$$

$$R_4(u', v') = -\frac{\Phi Z_1}{2\pi^2 Z_o} \sin[\frac{1}{2}(-\frac{\pi}{2} + ju') T(v', u')]$$

Taking into account the discussion of the system (C3)-(C4), the compactness of the kernels of the system (C8)-(C9) reduces to the compactness of the kernel:

$$\frac{2(-1)^{1/4}}{\sqrt{1-x} + j\sqrt{1+x}} \frac{B_{4r}(x, x')}{(1-x^2)^{1/4+\delta} (1-x'^2)^{1/2-\delta}}$$

This kernel is compact provided that:

$$|B_{4r}(x, x')|^2 = \left| \int_{-\infty}^{\infty} B_4(v(u(x), v') R_4(u(x'), v') dv' \right|^2 < \infty \quad -1 \leq x, x' \leq 1 \quad (C10)$$

We observe that  $\text{Max}\{B_4(v, v')\} < \infty$  for every value of  $-\infty \leq v, v' \leq \infty$ ; furthermore numerical simulations show that  $\int_{-\infty}^{\infty} R_4(u(x'), v') dv' = O[(1-x'^2)^{1/4}]$  for every  $-1 \leq x' \leq 1$ .

Whence, taking into account (C5), equation (C10) holds.

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